Kuwait University | Math 102 | March 27, 2008
Math. & Comp. Sci. Dept. | First Exam | Time: 90 minutes

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**Calculators and Mobile Phones are not allowed.**

1. Let $f(x) = e^x + e^{\tan^{-1} x}, \quad -\infty < x < \infty.$

   i) Show that $f^{-1}$ exists and find its domain.

   ii) Show that $P(2, 0)$ is on the graph of $f^{-1}$ and find the slope of the tangent line to the graph of $f^{-1}$ at $P(2, 0)$.

**(5 Points)**

2. Use logarithmic differentiation to find $\dfrac{dy}{dx}$ if $y = \dfrac{(\cosh x)^{\sec^{-1} x}}{\sqrt{e^x \ln|x|}}$.

**(4 Points)**

3. a) Show that: $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1.$

   b) Prove the identity: $\cos(2 \sin^{-1} x) = 1 - 2x^2$.

**(2 Points Each)**

4. Evaluate the following integrals:

   i) $\displaystyle\int \frac{1}{x(9 + (\ln x)^2)}\, dx.$

   ii) $\displaystyle\int 6^x\, 2^{-x}\, dx.$

   iii) $\displaystyle\int \frac{\tanh x}{\sqrt{4 \cosh^2 x - 1}}\, dx.$

   iv) $\displaystyle\int \frac{\sec x}{\sin x + 4 \cos x}\, dx.$

**(3 Points Each)**

**First Midterm Math102**, March 27, 2008

1. (a) $f'(x) = e^x + \frac{e^{\tan^{-1} x}}{1+x^2} > 0$ for all $x$, so $f$ is increasing. Therefore $f^{-1}$ exists.
   $\lim_{x\to-\infty}(e^x + e^{\tan^{-1} x}) = 0 + e^{-\pi/2} = e^{-\pi/2}$ and
   $\lim_{x\to\infty}(e^x + e^{\tan^{-1} x}) = \infty + e^{\pi/2} = \infty$, so $D_{f^{-1}} = R_f = (e^{-\pi/2}, \infty)$.

   (b) $f(0) = 2 \;\Rightarrow\; f^{-1}(2) = 0$, so $P(2, 0)$ is on the graph of $f^{-1}$.

2. $\ln y = \sec^{-1} \ln\cosh x - \frac{1}{2}x - \frac{1}{2}\ln\ln|x|$, so
   $\frac{1}{y}\frac{dy}{dx} = \left\{\sec^{-1} x \tanh x + \frac{\ln\cosh x}{x\sqrt{x^2-1}} - \frac{1}{2} - \frac{1}{2x\ln|x|}\right\}$ and
   $\frac{dy}{dx} = \left\{\sec^{-1} x \tanh x + \frac{\ln\cosh x}{x\sqrt{x^2-1}} - \frac{1}{2} - \frac{1}{2x\ln|x|}\right\} \frac{(\cosh x)^{\sec^{-1} x}}{\sqrt{e^x \ln|x|}}$

3. (a) $y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x \quad \Rightarrow \quad \cos y \cdot \frac{dy}{dx} = 1 \Rightarrow$
   $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$.

   (b) Let $\theta = \sin^{-1} x \Rightarrow \sin\theta = x$ and so
   $\cos(2\sin^{-1} x) = \cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 1 - 2x^2$.

4. (a) Let $u = \ln x$, then $\int \frac{dx}{x(9+(\ln x)^2)} = \int \frac{du}{9+u^2} = \frac{1}{3}\tan^{-1}(\frac{u}{3}) + C$
   $= \frac{1}{3}\tan^{-1}\left(\frac{\ln x}{3}\right) + C$

   (b) $\int 6^x 2^{-x} dx = \int 3^x 2^x 2^{-x} dx = \int 3^x dx = \frac{3^x}{\ln 3} + C$

   (c) Let $u = 2\cosh x$, then $\int \frac{\tanh x}{\sqrt{4\cosh^2 x - 1}} dx = \int \frac{2\sinh x}{2\cosh x\sqrt{4\cosh^2 x-1}} dx$
   $= \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C = \sec^{-1}(2\cosh x) + C$

   (d) $\int \frac{\sec x}{\sin x + 4\cos x} dx = \int \frac{\sec^2 x}{\sec x(\sin x + 4\cos x)} dx = \int \frac{\sec^2 x}{\tan x + 4} dx$
   $= \ln|\tan x + 4| + C$